Some Finitely Additive Versions of the Strong Law of Large Numbers

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1. Introduction.

Let X be a non-empty set with the discrete topology, $H = X^{\infty} = X \times X \times \ldots$ with the product topology, and $F(X)$ the set of all finitely additive probability measures defined on the class of all subsets of X. As defined by Dubins and Savage in [5], a strategy σ on H is a sequence $(\sigma_0, \sigma_1, \sigma_2, \ldots)$, where σ_0 is in $F(X)$, and, for each positive integer n, σ_n is a mapping from $X \times X \times \ldots \times X$ (n factors) to $F(X)$. For any positive integers n, any element $(x_1, \ldots, x_n)$ in X^n, $(x_1, \ldots, x_n)$ is called a partial history with length n. Suppose that σ is a strategy on H, $p = (x_1, \ldots, x_n)$ is a partial history with length n, then the conditional strategy given the partial history P with respect to the strategy σ is a strategy (written by $\sigma[p]$) on H defined by

(i) $(\sigma[p])_0 = \sigma_n(p) = \sigma_n(x_1, \ldots, x_n)$ i.e., $(\sigma[p])_0$ is just the finitely additive probability measure $\sigma_n(x_1, \ldots, x_n)$.

(ii) for any positive integer m ($m = 1, 2, \ldots$), $(\sigma[p])m$ is a mapping from X^m to $F(X)$ defined by $(\sigma[p])_m(x_1', x_2', \ldots, x_m') = \sigma_{n+m}(x_1, x_2, \ldots x_n, x_1', \ldots, x_m')$ for all $(x_1', x_2', \ldots, x_m')$ in X^m.

In [8], Purves and Sudderth call a strategy σ on H independent, if there exists a sequence $\{\gamma_n\}$ of finitely additive probability measures in $F(X)$ such that $\sigma_0 = \gamma_1$, and, for each positive integer n, for all n-tuple $(x_1, x_2, \ldots, x_n)$ in X^n, $\sigma_n(x_1, \ldots, x_n) = \gamma_{n+1}$ and they write $\gamma_1 \times \gamma_2 \times \ldots$ for σ. If, in addition, there exists a finitely additive probability measure γ in $F(X)$ such that $\gamma = \gamma_1 = \gamma_2 = \ldots$, then σ is said to be independent and identically distributed, and $\gamma \times \gamma \times \gamma \times \ldots$ is written for such a strategy.

In [8] Purves and Sudderth showed that if σ is a strategy on H, then there exists a field $a(\sigma)$ and a finitely additive probability measure (still denoted by σ) such that σ is defined on $a(\sigma)$ with some nice properties. Based on this result, we can consider $(H, a(\sigma), \sigma)$ as a finitely additive probability space and a standard theory of integration with respect to the finitely additive probability measure σ on the field $a(\sigma)$ is available (cf. [3]). Later we will use $\sigma(Y)$ to denote the integral of the function Y with respect to the strategy σ.

A sequence $\{Y_n\}$ of real-valued functions defined on H is called a sequence of coordinate mappings defined on H, if, for each positive integer n, the function Y_n depends only on the n^{th} coordinate. A sequence $\{Y_n\}$ of real-valued functions defined on H is called a sequence of identical coordinate mappings defined on H, if $\{Y_n\}$ is a sequence of coordinate mappings and, for any $1 \leq n < m < \infty$, $Y_n(h) = Y_m(h)$ whenever $h = (x_1, \ldots, x_n, \ldots, x_m, \ldots)$ in H and $x_n = x_m$.

In [8], Purves and Sudderth have shown that if σ is an independent strategy, $\{Y_n | \, n = 1, 2, \ldots\}$ is a sequence of coordinate mappings. $|Y_n(h)| \leq K < \infty$ for all $n = 1, 2, \ldots,$ for all $h \in H$, and $\sigma(Y_n) = 0$ for all $n = 1, 2, \ldots$. Then, the set $A = [h | \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0]$ has σ-measure one. In this paper, we show that the above result still holds without boundedness assumption (see Theorem 4-1 and Theorem 4-2 below). Furthermore, we will also show that "Kolmogorov's strong law of large numbers" holds for an independent, identically distributed strategy, and a sequence $\{Y_n | n = 1, 2, \ldots\}$ of identical, coordinate mappings (see Theorem 4-3 below).

2. Preliminary Definitions and Some Useful Lemmas.

Throughout this paper, X, H, $(H, a(\sigma), \sigma)$, and $\sigma(Y)$ are as defined in Section 1.

Definition 2-1.

A stopping rule τ on H is a mapping from H to the set of all positive integers such that if $h, h' \in H$, and h' agrees with h through the first $\tau(h)$ coordinates, then $\tau(h') = \tau(h)$. An incomplete stopping rule τ^* on H is a mapping from H to the set of all positive integers and ∞ such that if $h, h' \in H$, and h' agrees with h through the first $\tau(h)$ coordinates, then $\tau(h') = \tau(h)$.

Definition 2-2.

A subset K of H is said to be determined by a stopping rule τ on H, if, for any h in K and $h' \in H$ such that $\tau(h) = \tau(h')$, then h' must be in K.

The proof of Lemmas 2-1, 2-2, 2-3 is straightforward and the details are presented in [3].

Lemma 2-1.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H and Y is a real-valued function defined on H such that Y depends only on the K^{th} coordinate and $Y(h) = f(x_K)$ if $h = (x_1, x_2, \ldots x_K, \ldots) \in H$ where

f is a real-valued function defined on X. Then, we have

(i) f is γ_K-integrable if and only if Y is σ-integrable

(ii) Y is σ-integrable if and only if, for all $(x_1,\ldots,x_{K-1})$ in X^{K-1} $Yx_1,\ldots x_{K-1}$ is $\sigma[x_1,\ldots x_{K-1}]$ - integrable and the $\sigma[x_1,\ldots,x_{K-1}]$ - integral of the function $Yx_1,x_2,\ldots x_{K-1}$ is independent of $(x_1,\ldots x_{K-1})$.

Furthermore, we have

$\gamma_K(f) = \sigma[x_1,\ldots,x_{K-1}](Yx_1,\ldots x_{K-1}) = \sigma(Y)$ for all $(x_1,x_2,\ldots,x_{K-1})$ in X^{K-1}. Hence $\sigma(Y) = \int\ldots\int \sigma[x_1,\ldots,x_{K-1}](Yx_1,\ldots,x_{K-1})\, d\,\gamma_{K-1},\ldots,d\,\gamma_1$ whenever these integrals exist.

<u>Lemma 2-2</u>.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, $\{Y_n | n = 1, 2,\ldots\}$ is a sequence of coordinate mappings, and $1 \le r < \infty$. Suppose that $|Y_1|^r, |Y_2|^r,\ldots$ are σ-integrable. Then, for any $1 \le M < N < \infty$, $|\sum_{j=M}^{N} Y_j|^r$ is σ-integrable. Furthermore, for all $(x_1,\ldots,x_{M-1})$ in X^{M-1}, $\sigma[x_1,\ldots,x_{M-1}](|\sum_{j=M}^{N} Y_j|^r(x_1,\ldots x_{M-1}) = \sigma(|\sum_{j=M}^{N} Y_j|^r)$.

<u>Lemma 2-3</u>.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, and, Y, Z are two real-valued functions defined on H such that Y depends only on the first p coordinates and Z depends only on the first $p+m$ to $p+m+\ell$ coordinates where $1 \le p < \infty$, $1 \le m < \infty$, $1 \le \ell < \infty$. Then, if Y and Z are σ-integrable, YZ is σ-integrable and $\sigma(YZ) = \sigma(Y)\,\sigma(Z)$.

Lemma 2-4.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \dots$ is an independent strategy on H, and $\{Y_j | j = 1,2,\dots\}$ is a sequence of coordinate mappings. Suppose that $\sigma(Y_j) = 0$ $\sigma(|Y_j|^{2r}) < \infty$ for all $j = 1, 2, \dots$ where r is a positive constant and $r \geq 1$. Then there exists a positive constant A such that for all $0 \leq m < n < \infty$, $\sigma(|\sum_{j=m+1}^{n} Y_j|^{2r}) \leq A(n-m)^{r-1} \sum_{j=m+1}^{n} \sigma(|Y_j|^{2r})$ and the constant A depends only on r and not on m, n.

Proof: The proof of this lemma is essentially the same as the one in the conventional theory of probability (see [6]) and it is too lengthy to present here.

Lemma 2-5.

Suppose that $a_1, a_2, \dots$ are real numbers such that $\sum_{j=1}^{n} a_j$ converges as $n \to \infty$, and suppose that $b_1, b_2, \dots$ are positive real numbers such that $0 < b_1 \leq b_2 \leq, \dots,$ $\lim_{n\to\infty} b_n = \infty$. Then $\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} a_j b_j = 0$.

Proof: See page 51 of [1].

Lemma 2-6.

Suppose that $\{a_n\}$ is a sequence of non-negative real-numbers, and $\{b_n\}$ is a sequence of positive real-numbers such that $0 < b_1 \leq b_2 \leq, \dots,$ $\lim_{n\to\infty} b_n = \infty$ and $\sum_{j=1}^{\infty} \frac{a_j}{b_j} < \infty$. Then there exists a sequence $\{C_n | n=0,1,2,3,\dots\}$, of positive real numbers satisfying

(i) $\lim_{n\to\infty} \left(\frac{C_n}{b_{2^n}} \sum_{j=1}^{2^n} a_j\right) = 0$

(ii) $1 \leq C_n \leq C_{n+1} \leq C_n + 1$ for all $n = 0, 1, 2, \dots$ and $\lim_{n\to\infty} C_n = \infty$

(iii) $C_0 \frac{a_1}{b_1} + \sum_{K=1}^{\infty} C_K\left(\sum_{2^{K-1}<n\leq 2^K} \frac{a_n}{b_n}\right) < \infty$.

<u>Proof</u>: For each $n = 1, 2, 3, \ldots$, let $f(n) = \frac{1}{b_{2^n}} \sum_{j=1}^{2^n} a_j$. By Lemma 2-5, $\lim_{n\to\infty} f(n) = 0$. Hence, there exists a sequence $\{N_j \mid j = 1, 2, \ldots\}$ of strictly increasing positive integers such that $f(n) \leq \frac{1}{j^2}$ if $n \geq N_j$ for all $j = 1, 2, \ldots$.

Define $x_n = j$ if $N_j \leq n < N_{j+1}$ $j = 1, 2, \ldots$

$= 1$ if $0 \leq n < N_1$.

Then $x_n f(n) \leq \frac{1}{j}$ if $n \geq N_j$ $j = 1, 2, \ldots$. Since $N_j < \infty$ for all $j = 1, 2, \ldots$ and $\lim_{j\to\infty} N_j = \infty$, $\lim_{n\to\infty} x_n f(n) = 0$.

By assumption, $\sum_{j=1}^{\infty} \frac{a_j}{b_j} < \infty$, $\frac{a_j}{b_j} \geq 0$ $j = 1, 2,$ we can and do assume $\sum_{j=1}^{\infty} \frac{a_j}{b_j} \leq 1$. Let $K_0 = 0$, and for each $i = 1, 2, \ldots,$ let

$K_i = \max\{K_{i-1} + 1,\ \inf\{j \mid \sum_{n=j}^{\infty} \frac{a_n}{b_n} \leq \frac{1}{2^i}\}\}$. It is easy to see that $1 \leq K_1 < K_2 < \ldots < \infty$ and $\lim_{j\to\infty} K_j = \infty$. Now, for each $i = 1, 2, 3, \ldots,$ each $j = 1, 2, \ldots$ define

$$y_j^i = \begin{cases} 0 & \text{if } j < K_i \\ 1 & \text{if } j \geq K_i . \end{cases}$$

Then $\sum_{K=1}^{\infty} y_K^i \frac{a_K}{b_K} = \sum_{K=K_i}^{\infty} \frac{a_K}{b_K} \leq \frac{1}{2^i}$ for all $i = 1, 2, \ldots$.

Define $y_K = 1$, $K = 1, 2, \ldots, \ldots, K_1 - 1$

$= \sum_{i=1}^{\infty} y_K^i$ for all $K \geq K_1$

so $y_K \geq i$ if $K \geq K_i$ for all $i = 1, 2, 3\ldots$.

Since $K_i < \infty$ for all $i = 1, 2, 3 \ldots$, $\lim_{K \to \infty} y_K = \infty$.

By the definitions of $\{y_K^i | i = 1, 2, \ldots K = 1, 2, \ldots\}$ and $\{y_K | K = 1, 2, 3 \ldots\}$,

We have

$$y_K \geq 1, \quad y_K \leq y_{K+1} \quad \text{for all } K = 1, 2, \ldots, \quad \text{and}$$

$$\sum_{K=1}^{\infty} y_K \frac{a_K}{b_K} = \sum_{K=1}^{K_1 - 1} y_K \frac{a_K}{b_K} + \sum_{K=K_1}^{\infty} y_K \frac{a_K}{b_K} = \sum_{K=1}^{K_1 - 1} \frac{a_K}{b_K} + \sum_{K=K_1}^{\infty} \sum_{i=1}^{\infty} y_K^i \frac{a_K}{b_K}$$

$$= \sum_{K=1}^{K_1 - 1} \frac{a_K}{b_K} + \sum_{i=1}^{\infty} \sum_{K=K_1}^{\infty} y_K^i \frac{a_K}{b_K} \leq \sum_{K=1}^{K_1 - 1} \frac{a_K}{b_K} + \sum_{i=1}^{\infty} \sum_{K=1}^{\infty} y_K^i \frac{a_K}{b_K}$$

$$\leq \sum_{K=1}^{K_1 - 1} \frac{a_K}{b_K} + \sum_{i=1}^{\infty} \frac{1}{2^i} \leq 2 \quad \left(\text{since } \sum_{K=1}^{\infty} y_K^i \frac{a_K}{b_K} \leq \frac{1}{2^j} \text{ and } \sum_{K=1}^{\infty} \frac{a_K}{b_K} \leq 1\right)$$

Now, define $Z_0 = 1$, $Z_K = y_{2^{K-1}}$ $K = 1, 2, 3 \ldots$, then we have

(i) $1 \leq Z_K \leq Z_{K+1}$ for all $K = 0, 1, 2, \ldots$ and $\lim_{K \to \infty} Z_K = \infty$.

(ii) $Z_0 \frac{a_1}{b_1} + \sum_{k=1}^{\infty} Z_K \left(\sum_{2^{K-1} < n \leq 2^K} \frac{a_n}{b_n} \right) < \infty$.

Let $c_n^* = \min(x_n, Z_n)$ for all $n = 0, 1, 2, \ldots$. Since $x_n \leq x_{n+1}$, $Z_n \leq Z_{n+1}$ for all $n = 0, 1, 2, \ldots$ and $\lim_{n \to \infty} x_n = \infty$, $\lim_{n \to \infty} Z_n = \infty$, we have

1^0) $1 \leq c_n^* \leq c_{n+1}^*$ for all $n = 0, 1, 2, \ldots$, and $\lim_{n \to \infty} c_n^* = \infty$

2^0) $\lim_{n \to \infty} c_n^* f(n) = 0$

3^0) $c_0^* \frac{a_1}{b_1} + \sum_{K=1}^{\infty} c_K^* \left(\sum_{2^{K-1} < n \leq 2^K} \frac{a_n}{b_n} \right) < \infty$.

Finally, we inductively define $C_K (K = 0, 1, 2, \ldots)$ by letting $C_0 = 1$ and for $n \geq 1$, letting $C_n = \min\{C_n^*, C_{n-1} + 1\}$.

Then, we have

(i) $C_n \leq C_n^*$ for all $n = 0, 1, 2, \ldots$

(ii) $1 \leq C_n \leq C_{n+1} \leq C_n + 1$ for all $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} C_n = \infty$

(iii) $\lim_{n\to\infty} C_n f(n) = \lim_{n\to\infty} \{\frac{C_n}{b_{2^n}} \sum_{j=1}^{2^n} a_j\} = 0$

(iv) $C_0 \frac{a_1}{b_1} + \sum_{K=1}^{\infty} C_K (\sum_{2^{K-1} < n \leq 2^K} \frac{a_n}{b_n}) < \infty$.

All, except the statements $\lim_{n\to\infty} C_n = \infty$, and (iv), are obvious, we prove these two statements as follows. First, suppose that there exists a sequence $\{n_j\}$ of positive integers such that

$$C_{n_j} = C_{n_j}^* \quad \text{for all } j = 1, 2, \ldots, \text{ and } \lim_{j\to\infty} n_j = \infty$$

then, by the fact $C_n \leq C_{n+1}$ for all $n = 0, 1, 2, \ldots$, $C_n \to \infty$ as $n \to \infty$. Next, suppose that there exists a positive integer N such that $C_n \neq C_n^*$ for all $n > N$, then $C_{N+K} = C_N + K$, hence $\lim_{n\to\infty} C_n = \infty$.

Since $$C_0 \frac{a_1}{b_1} + \sum_{K=1}^{\infty} C_K (\sum_{2^{K-1} < n \leq 2^K} \frac{a_n}{b_n})$$

$$\leq C_0^* \frac{a_1}{b_1} + \sum_{K=1}^{\infty} C_K^* (\sum_{2^{K-1} < n \leq 2^K} \frac{a_n}{b_n}) < \infty .$$

The proof of Lemma 2-6 is, now, complete.

Lemma 2-7.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \dots$ is an independent strategy on H, and $\{Y_n | n = 1, 2, \dots\}$ is a sequence of coordinate mappings defined on H. Let $S_0 = 0$ and, for each $n = 1, 2, 3, \dots$ $S_n = \sum_{j=1}^{n} Y_j$. Then, if $\epsilon > 0$, $\delta > 0$, M, N are two integers such that $0 \leq M < N < \infty$ and

$$\max_{M \leq n < N} \sigma([h| \; |S_N(h) - S_M(h)| > \epsilon]) \leq \delta ,$$

we have $\sigma([h| \max_{M < n \leq N} |S_n(h) - S_M(h)| > 2 \epsilon])$

$$\leq \frac{1}{1-\delta} \sigma([h| \; |S_N(h) - S_M(h)| > \epsilon]) .$$

Proof: This result does not seem to have been stated before. The proof is essentially the same as the one in the conventional theory of probability (see page 45 of [1]) and we omit it.

3. Borel Cantelli Lemmas.

In this section, we will state some finitely additive versions of Borel Cantelli Lemmas. Theorem 3-1 is a central result for proving strong laws of large numbers. Except Lemma 3-1, Theorem 3-1, and Theorem 3-3, all the material of this section and more is available in [7] and [8]. For the sake of reference, we state these results in this section.

Let $\{K_n\}$ be a sequence of clopen subsets of H, and $\{\tau_n\}$ be a strictly increasing sequence of stopping rules defined on H such that, for each $n = 1, 2, 3 \ldots,$ K_n is determined by time τ_n. From these two sequences $\{K_n\}$ and $\{\tau_n\}$, we define, for each positive integer, each element $h = (x_1, x_2, \ldots)$ in H, a partial history $q_n(h) = P_{\tau_n}(h) = (x_1, x_2, \ldots x_{\tau_n(h)})$ and a clopen subset $K_{n+1}\, q_n(h) = [h' | h' = (x_1', x_2', \ldots x_n' \ldots) \in H$ and $(x_1, x_2, \ldots x_{\tau_n(h)}, x_1', x_2', \ldots) \in K_{n+1}]$. The following is an important lemma for the σ-probability measure of countable intersections of clopen subsets of H.

Lemma 3-1.

Suppose that $\{K_n\}$, $\{\tau_n\}$, $\{q_n(h) | h \in H\}$, $\{K_{n+1} q_n(h) | h \in H\}$ are as defined above. Let $g_n(h) = \sigma[q_{n-1}(h)](K_n q_{n-1}(h))$ $n = 2, 3, 4, \ldots$ and $g_1(h) = \chi_{K_1}(h)$. Suppose that $\{\alpha_n\}$ is a sequence of positive real numbers such that $0 \le \alpha_n \le 1$, for all $n = 1, 2, \ldots$. Then, if $K_1 \neq \emptyset$, and for each $n = 1, 2, 3, \ldots,$ each h in $\bigcap_{i=1}^{n} K_i$, $K_{n+1} q_n(h) \neq \emptyset$, and if, for each $n = 1, 2, 3, \ldots,$ each h^* in $\bigcap_{j=1}^{n-1} K_j$

$$\sigma[q_{n-1}(h^*)]([\bigcap_{j=1}^{n} K_j - \{h | g_{n+1}(h) \ge \alpha_{n+1}\}]\, q_{n-1}(h^*)) = 0 ,$$

we will have $\sigma(\bigcap_{j=1}^{\infty} K_j) \geq \sigma(K_1)\prod_{j=2}^{\infty} \alpha_j$, if, in addition $\sigma(K_1) \geq \alpha_1$, then $\sigma(\bigcap_{j=1}^{\infty} K_j) \geq \prod_{j=1}^{\infty} \alpha_j$.

Proof: The proof is essentially the same as that of Theorem 6-1 in [8] and we omit it.

Theorem 3-1.

Let $\{K_n\}$, $\{\tau_n\}$, $\{q_n(h) | h \in H\}$, $\{K_{n+1}q_n(h) | h \in H\}$, and $\{\alpha_n\}$ be defined above. If

(i) these exist a strictly increasing sequence $\{n_j\}$ of positive integers such that $\lim_{j \to \infty} \sigma(K_{n_j}^c) = 1$

(ii) for each $j = 1, 2, 3, \ldots$, $m = 0, 1, 2, \ldots$, and each $h \in \bigcap_{\ell=0}^{m} K_{n_j+\ell}^c$, $\sigma[q_{n_j+m}(h)](K_{n_j+m+1}^c q_{n_j+m}(h))$

$\geq 1 - \alpha_{n_j+m+1}(n_j)$ and $\lim_{j \to \infty} \prod_{m=0}^{\infty} (1 - \alpha_{n_j+m+1}(n_j)) = 1$

where, for all $j = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$,

$0 \leq \alpha_{n_j+m+1}(n_j) \leq 1$ and $\alpha_{n_j+m+1}(n_j)$ is a constant which may depend on n_j.

Then $\sigma(K_n \text{ i, } O(n)) = \sigma(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} K_n) = 0$.

Proof: Notice that $[K_n \text{ i, } O(n)] = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} K_n$. So, $[K_n \text{ i}, O] \leq \bigcup_{\ell=n_j}^{\infty} K_\ell$ for all $j = 1, 2, \ldots$. By Lemma 3-1, $\sigma(\bigcap_{\ell=n_j}^{\infty} K_\ell^c) \geq \sigma(K_{n_j}^c)\prod_{m=0}^{\infty}(1 - \alpha_{n_j+m+1}(n_j))$.

Hence, $\lim_{j \to \infty} \sigma(\bigcap_{\ell=n_j}^{\infty} K_\ell^c) \geq \lim_{j \to \infty}\{\sigma(K_{n_j}^c)\prod_{m=0}^{\infty} (1 - \alpha_{n_j+m+1}(n_j))\}$

i.e, $\lim_{j \to \infty} \sigma(\bigcup_{\ell=n_j}^{\infty} K_\ell) \leq 1 - \lim_{j \to \infty} \{\sigma(K_{n_j}^c) \prod_{m=0}^{\infty} (1 - \alpha_{n_j+m+1}(n_j))\} = 0$.

Therefore $\sigma[K_m\ i,\ O(m)] \le \lim_{j\to\infty} \sigma(\bigcup_{\ell=n_j}^{\infty} K_\ell) = 0$.

Corollary 3-1.

Let $\sigma = r_1 \times r_2 \times \ldots$ be an independent strategy on H, and $N_1, N_2, \ldots$ be positive integers. Suppose that $A_j \subset X^{N_j}$ (N_j factors) for each $j = 1, 2, \ldots$. Let $r_1 = 1$, $r_j = 1 + \sum_{i=1}^{j-1} N_i$ $j = 1, 2, 3, 4, \ldots$ $S_j = \sum_{i=1}^{j} N_i$ $j = 1, 2, \ldots$, $K_n = X^{r_n - 1} \times A_n \times H$ $n = 1, 2, \ldots$.

Suppose that $\sum_{n=1}^{\infty} \sigma(K_n) < \infty$, then $\sigma[K_n\ i, O(n)] = 0$.

Proof: See page 36 of [7].

Theorem 3-2.

Suppose that, for each $n = 1, 2, \ldots$, and for all h in H, $\sigma[q_n(h)](K_{n+1} q_n(h)) \ge \alpha_{n+1}$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\sigma(K_n\ i, O(n)) = 1$.

Proof: See pages 36-37 of [7].

Corollary 3-2.

Let σ, $\{K_n\}$ be as in Corollary 3-1 and $\sum_{n=1}^{\infty} \sigma(K_n) = \infty$. Then $\sigma(K_n\ i, O(n)) = 1$.

Proof: See page 37 of [7].

Theorem 3-3.

Let σ and $\{K_n\}$ be as in Corollary 3-1. Then

(i) $\sum_{n=1}^{\infty} \sigma(K_n) < \infty$ if and only if $\sigma(K_n\ i,\ O(n)) = 0$

(ii) $\sum_{n=1}^{\infty} \sigma(K_n) = \infty$ if and only if $\sigma(K_n\ i,\ O(n)) = 1$.

Proof: By Corollary 3-1, Corollary 3-2, and "Kolmogorov zero-one law" for an independent strategy. For the proof of Kolmogorov zero-one law, see pages 49-50 of [8].

4. Strong Laws of Large Numbers.

Now, we are in the position to prove strong laws of large numbers.

Theorem 4-1.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, and $\{Y_n | n = 1, 2, \ldots\}$ is a sequence of coordinate mappings on H. Suppose that $\sigma(Y_n) = 0$ for all $n = 1, 2, \ldots$, and $\sigma(|Y_n|^{2r}) < \infty$ for all $n = 1, 2, \ldots$, where r is a constant such that $1 \le r < \infty$. If

$\sum_{n=1}^{\infty} \frac{\sigma(|Y_n|^{2r})}{n^{1+r}} < \infty$, then the set $A = [h | \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0]$ has σ-measure 1.

Proof: Choose, as is possible by Lemma 2-6, a sequence $\{C_n\}$ of positive real numbers such that

$$\text{(i)} \quad \lim_{n \to \infty} [C_n \cdot \frac{1}{(2^n)^{1+r}} \sum_{j=1}^{2^n} \sigma(|Y_j|^{2r})] = 0$$

$$\text{(ii)} \quad 1 \le C_n \le C_n + 1 \text{ for all } n = 0,1,2,3,\ldots \text{ and } \lim_{n \to \infty} C_n = \infty$$

$$\text{(iii)} \quad C_0 \frac{\sigma(|Y_1|^{2r})}{1^{1+r}} + \sum_{K=1}^{\infty} C_K \left(\sum_{2^{K-1} < n \le 2^K} \frac{\sigma(|Y_n|^{2r})}{n^{1+r}} \right) < \infty .$$

Now, we prove Theorem 4-1 in the following three steps. (For $n = 1, 2, 3 \ldots$, let $S_n = \sum_{j=1}^{n} Y_j$, $S_0 = 0$ and for $m = 0, 1, 2, \ldots$, $n = 1, 2, 3, \ldots$ $m < n$, let $S_{m,n} = S_n - S_m$.)

Step 1.

$$\sigma([h | h \in H, \lim_{n \to \infty} \frac{S_{2^n}(h)}{2^n} = 0]) = 1 .$$

To see this, let $K_n = [h \mid \frac{1}{2^n} \mid \sum_{j=1}^{2^n} Y_j(h) \mid > \frac{1}{4r\sqrt{C_n}}]$ $n = 1, 2, 3, \ldots$

By Chehyshev's inequality , we have

$$\sigma(K_n) \le \frac{1}{\left(\frac{1}{4r\sqrt{C_n}}\right)^{2r}} \sigma\left(\left|\frac{S_{2^n}}{2^n}\right|^{2r}\right)$$

$$= \frac{\sqrt{C_n}}{(2^n)^{2r}} \sigma(|S_{2^n}|^{2r}).$$

By Lemma 2-4, $\sigma(|S_{2^n}|^{2r}) \le A(2^n)^{r-1} \sum_{j=1}^{2^n} \sigma(|Y_j|^{2r})$ where A is a constant, $0 < A < \infty$.

Hence $\sigma(K_n) \le \frac{AC_n}{(2^n)^{1+r}} \sum_{j=1}^{2^n} \sigma(|Y_j|^{2r})$ (since $C_n \ge 1$). By (i) above, we have

$$\lim_{n \to \infty} \sigma(K_n^c) = 1 - \lim_{n \to \infty} (K_n) = 1 .$$

For each $n \ge 1$, define a stopping rule $\tau_n = 2^n$ on H, a partial history $q_{n+m}(h) = (x_1, x_2, \ldots, x_{2^{n+m}})$ if $h = (x_1, x_2, \ldots, x_{2^{n+m}}, \ldots) \in H$. For each $n = 1, 2, \ldots,$ $m = 0, 1, 2, \ldots,$ h in H define

$$A_{n+m+1}(h) = \{h' \mid h' \in H \ |S_{2^{n+m+1}}(q_{n+m}(h)\, h') - S_{2^{n+m}}(q_{n+m}(h)\, h')|$$

$$\le \left(\frac{2^{n+m+1}}{4r\sqrt{C_{n+m+1}}} - \frac{2^{n+m}}{4r\sqrt{C_{n+m}}}\right)\} .$$

It is easy to check that, for each $h \in \bigcap_{\ell=0}^{m} K^c_{\ell+m}$, $K^c_{n+m+1} q_{n+m}(h) \supseteq A_{n+m+1}(h)$.

Hence,

$$\sigma[q_{n+m}(h)](K^c_{n+m+1} q_{n+m}(h))$$

$$\ge \sigma[q_{n+m}(h)](A_{n+m+1}(h))$$

$$\geq 1 - \frac{\sigma[q_{n+m}(h)]([|S_{2^{n+m+1}} - S_{2^{n+m}}|q_{n+m}(h)]^{2r})}{\left(\frac{2^{n+m+1}}{4r\sqrt{C_{n+m+1}}} - \frac{2^{n+m}}{4r\sqrt{C_{n+m}}}\right)^{2r}} .$$

By Lemma 2-2, the last expression is equal to

$$1 - \frac{\sigma(|S_{2^{n+m+1}} - S_{2^{n+m}}|^{2r})}{\left(\frac{2^{n+m+1}}{4r\sqrt{C_{n+m+1}}} - \frac{2^{n+m}}{4r\sqrt{C_{n+m}}}\right)^{2r}} .$$

Hence $\sigma[q_{n+m}(h)](K^c_{n+m+1}\ q_{n+m}(h))$

$$\geq 1 - \frac{A(2^{n+m})^{r-1} \sum_{j=2^{n+m}+1}^{2^{n+m+1}} \sigma(|Y_j|^{2r})}{(2^{n+m+1})^{2r}\left(\frac{1}{4r\sqrt{C_{n+m+1}}} - \frac{1}{2\,4r\sqrt{C_{n+m}}}\right)^{2r}}$$

$$= 1 - \frac{A\sqrt{C_{n+m}}\sqrt{C_{n+m+1}} \sum_{j=2^{n+m}+1}^{2^{n+m+1}} \sigma(|Y_j|^{2r})}{(2^{n+m})^{2r-r+1}2^{2r}(4r\sqrt{C_{n+m}} - \frac{1}{2}\,4r\sqrt{C_{n+m+1}})^{2r}} .$$

Since $1 \leq C_{n+m} \leq C_{n+m+1} \leq C_{n+m} + 1$ and $\lim_{n\to\infty} C_n = \infty$, there exists an N (positive integer) such that, for all $n \geq N$ $(4r\sqrt{C_{n+m}} - \frac{1}{2}\,4r\sqrt{C_{n+m+1}}) \geq \frac{1}{2}$.
Therefore, if $n \geq N$, $m \geq 0$, $h \in \bigcap_{\ell=0}^{m} K^c_{n+\ell}$,

$$\sigma[q_{n+m+1}(h)](K^c_{n+m+1}q_{n+m}(h))$$

$$\geq 1 - \frac{A\, C_{n+m+1}}{(2^{m+n+1})^{1+r}\, 2^{r-1}(\frac{1}{2})^{2r}} \sum_{j=2^{n+m}+1}^{2^{n+m+1}} \sigma(|Y_j|^{2r}) .$$

By (iii) above

$$\sum_{K=1}^{\infty} C_K \sum_{2^{K-1}<n\leq 2^K} \frac{\sigma(|Y_n|^{2r})}{n^{1+r}} < \infty$$

implies that

$$\sum_{K=N}^{\infty} \frac{A\, C_{K+1}}{(2^{K+1})^{1+r}\, 2^{r-1}(\frac{1}{2})^{2r}} \sum_{j=2^K+1}^{2^{K+1}} \sigma(|Y_j|^{2r}) < \infty$$

and it is equivalent to

$$\lim_{n\to\infty} \prod_{m=0}^{\infty} \left(1 - \frac{A\, C_{n+m+1}}{(2^{n+m+1})^{1+r}\, 2^{r-1}(\frac{1}{2})^{2r}} \sum_{j=2^{n+m}+1}^{2^{n+m+1}} \sigma(|Y_j|^{2r})\right) = 1 .$$

By Theorem 3-1, we have

$$\sigma([K_n \ i, \ O(n)]) = 0 \quad \text{i.e.,} \quad \sigma([K_n \ i, \ O(n)]^c) = 1 .$$

But the set $[h|\ h \in H \ \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]$ contains the set $[K_n \ i, \ O(n)]^c$.

Hence $\sigma([h|h \in H_n \lim_{n\to\infty} \frac{S_{2^n}(h)}{2^n} = 0]) = 1$.

<u>Step 2</u>.

$$\sigma([h|h \in H, \ \lim_{n\to\infty} \frac{\max_{2^n<K\leq 2^{n+1}} |S_K(h) - S_{2^n}(h)|}{2^n} = 0]) = 1 .$$

To see this, let $D_n = \max_{2^n<K\leq 2^{n+1}} |S_K - S_{2^n}| \quad n = 1, 2, 3\ldots$.

Since, by (iii) above,

$$C_0 \frac{\sigma(|Y_1|^{2r})}{1^{1+r}} + \sum_{K=1}^{\infty} C_K \sum_{2^{K-1}<n\le 2^K} \frac{\sigma(|Y_n|^{2r})}{n^{1+r}} < \infty ,$$

there is a positive integer N, such that, for all $n \ge N$,

$$A \sum_{K=n}^{\infty} C_K \sum_{2^{K-1}<j\le 2^K} \frac{\sigma(|Y_j|^{2r})}{j^{1+r}} \le \frac{1}{4 \times 2^{1+r}}$$

where A is the constant in the proof of step 1. Now, if $n \ge N$, then

$$\sigma(|S_{2^{n+1}} - S_{2^K}| > \frac{2^n}{2r\sqrt{C_n}}) \le \frac{C_n}{(2^n)^{2r}} \sigma([|S_{2^{n+1}} - S_K|^{2r}])$$

for all K such that $2^n \le K \le 2^{n+1}$

$$\le \frac{C_n}{(2^n)^{2r}} \cdot A(2^{n+1} - K)^{r-1} \sum_{j=K+1}^{2^{n+1}} \sigma(|Y_j|^{2r})$$

$$\le \frac{C_n}{(2^n)^{2r}} \cdot A(2^{n+1} - 2^n)^{r-1} \sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r}) .$$

Hence

$$\max_{2^n\le K\le 2^{n+1}} \sigma([|S_{2^{n+1}} - S_K| > \frac{2^n}{2r\sqrt{C_n}}])$$

$$\le \frac{C_n \cdot A}{(2^n)^{1+r}} \sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r}) \le \frac{1}{4} .$$

By Lemma 2-7, for all $n \ge N$

$$\sigma([\max_{2^n<K\le 2^{n+1}} |S_K - S_{2^n}| > \frac{2 \cdot 2^n}{2r\sqrt{C_n}}])$$

$$\leq \frac{1}{1 - \frac{1}{4}} \sigma([|S_{2^{n+1}} - S_{2^n}| > \frac{2^n}{2r\sqrt{C_n}}])$$

$$\leq \frac{4}{3} \cdot A \cdot \frac{C_n}{(2^n)^{2r}} (2^n)^{r-1} \sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r})$$

$$= \frac{4AC_n}{3} \cdot \frac{\sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r})}{(2^n)^{1+r}} \leq \frac{2^{3+r}AC_n}{3} \times \frac{\sum_{j=2^n+1}^{2^{n+1}} \sigma(|Y_j|^{2r})}{(2^{n+1})^{1+r}} .$$

Set $L_n = [D_n > \frac{2 \cdot 2^n}{2r\sqrt{C_n}}]$ then, if $n \geq N$,

$$\sigma(L_n) \leq \frac{2^{3+r}AC_n}{3} \sum_{j=2^n+1}^{2^{n+1}} \frac{\sigma(|Y_j|^{2r})}{j^{1+r}}$$

$$\leq \frac{2^{3+r}A}{3} C_{n+1} \sum_{j=2^n+1}^{2^{n+1}} \frac{\sigma(|Y_j|^{2r})}{j^{1+r}} .$$

Hence

$$\sum_{n=1}^{\infty} \sigma(L_n) \leq (N - 1) + \sum_{n=N}^{\infty} \sigma(L_n)$$

$$\leq (N - 1) + A \cdot \frac{2^{3+r}}{3} \sum_{n=N}^{\infty} C_n \sum_{j=2^n+1}^{2^{n+1}} \frac{\sigma(|Y_j|^{2r})}{j^{1+r}}$$

$$\leq (N - 1) + A \cdot \frac{2^{3+r}}{3} \sum_{n=N}^{\infty} C_{n+1} \sum_{j=2^n+1}^{2^{n+1}} \frac{\sigma(|Y_j|^{2r})}{j^{1+r}} < \infty$$

(by (iii) above).

By Corollary 3-1, $\sigma([L_n\ i,\ O(n)]^c) = 1$ and notice that the set $[h|\ h \in H,\ \lim_{n\to\infty} \frac{D_n}{2^n} = 0]$ contains the set $[L_n\ i,\ O(n)]^c$. Therefore

$$\sigma([h|\ \lim_{n\to\infty} \frac{\max_{2^n < K \leq 2^{n+1}} |S_K(h) - S_{2^n}(h)|}{2^n} = 0]) = 1$$

Step 3.

$$\sigma([h|\ \lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1 .$$

To see this, let us define $m(n)$ as the integer such that $2^{m(n)} \leq n < 2^{m(n)+1}$ $n = 2, 3, \ldots$. It is easy to check that

$$[h|\lim_{n\to\infty} \frac{S_n(h)}{n} = 0] \supseteq [h|\lim_{m(n)\to\infty} \frac{D_{m(n)}(h)}{2^{m(n)}} = 0] \cap [h|\lim_{m(n)\to\infty} \frac{|S_{2^{m(n)}}(h)|}{2^{m(n)}} = 0]$$

and therefore,

$$\sigma([h|\ \lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1 .$$

The proof of Theorem 4-1, now, is complete.

Corollary 4-1.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ in an independent strategy on H, $\{Y_n | n = 1, 2, \ldots\}$ is a sequence of coordinate mappings defined on H. Suppose that $\sigma(Y_n) = 0$, $\sigma(|Y_n|^2) < \infty$ for all $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \frac{\sigma(|Y_n|^2)}{n^2} < \infty$. Then $\sigma([h|\lim_{n\to\infty} \frac{S_n(h)}{n} = 0]) = 1$.

Proof: This is a special case of Theorem 4-1 when $r = 1$, and also a special case of Theorem 4-2 when $a_n = n$, $n = 1, 2, 3, \ldots$.

Theorem 4-2.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times, \ldots$ is an independent strategy on H and $\{Y_n | n = 1, 2, \ldots\}$ is a sequence of coordinate mappings on H such that $\sigma(Y_n) = 0$, $\sigma(Y_n^2) < \infty$, for all $n = 1, 2, \ldots$. Suppose that $a_1, a_2, \ldots$ are positive real numbers such that $0 < a_1 \le a_2 \le \ldots$ and $\lim_{n \to \infty} a_n = \infty$. Then, if

$$\sum_{n=1}^{\infty} \frac{\sigma(|Y_n|^2)}{a_n^2} < \infty, \quad \sigma([h| \lim_{n \to \infty} \frac{S_n(h)}{a_n} = 0]) = 1$$

Proof: For each $n = 1, 2, \ldots,$ let $Y_n^* = \frac{Y_n}{a_n}$, then $\sigma(Y_n^*) = 0$, $\sigma(Y_n^{*2}) = \frac{\sigma(Y_n^{*2})}{a_n^2}$ for all $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \sigma(Y_n^{*2}) < \infty$. For each $j = 1, 2, 3 \ldots,$ let $\epsilon_j = \frac{1}{(1+j)^{1+\alpha}}$ $\alpha > 0$. Define, $N_0 = 0$, and

$$N_j = \max\{N_{j-1} + 1, \ \inf\{K| \sum_{j=K}^{\infty} \sigma(Y_n^{*2}) \le \epsilon_j^3\}\}$$

$j = 1, 2, 3, \ldots$. Since $\sum_{j=1}^{\infty} \sigma(Y_n^{*2}) < \infty$, $1 \le N_1 < N_2 < \ldots < \infty$, and $\lim_{n \to \infty} N_j = \infty$. For each $j = 1, 2, 3 \ldots,$ define

$$D_j = \max_{N_j < \ell \le N_{j+1}} |\sum_{K=N_j+1}^{\ell} Y_K^*|, \quad L_j = [h| \ |D_j(h)| > 2\,\epsilon_j] .$$

Then, for $N_j < \ell \le N_{j+1}$,

$$\sigma([|\sum_{K=\ell}^{N_{j+1}} Y_K^*| > \epsilon_j])$$

$$\le \frac{1}{\epsilon_j^2} \sigma(|\sum_{k=\ell}^{N_{j+1}} Y_K^*|^2) = \frac{1}{\epsilon_j^2} \sum_{k=\ell}^{N_{j+1}} \sigma(|Y_k^*|^2)$$

$$\leq \frac{1}{\epsilon_j^2} \sum_{k=N_j+1}^{\infty} \sigma(|Y_k^*|^2) \leq \frac{1}{\epsilon_j^2} \epsilon_j^3 = \epsilon_j .$$

Since $\epsilon_j \leq \frac{1}{2^{1+\alpha}} < \frac{1}{2} \quad \forall\, j = 1, 2, \ldots,$ we have, by Lemma 2-7,

$$\sigma([\max_{N_j < \ell \leq N_{j+1}} |\sum_{i=N_j+1}^{\ell} Y_i^*| > 2\,\epsilon_j]) = \sigma(L_j)$$

$$\leq \frac{1}{1-\epsilon_j} \sigma(|\sum_{j=N_j+1}^{N_{j+1}} Y_j^*| > \epsilon_j) \leq \frac{\epsilon_j}{1-\epsilon_j} < 2\epsilon_j .$$

Hence $\sum_{j=1}^{\infty} \sigma(L_j) < \sum_{j=1}^{\infty} 2\epsilon_j < \infty$.

By Corollary 3-1, we have $\sigma([L_j \text{ i, } O(j)]^c) = 1$. Notice that the set $[h| \lim_{n\to\infty} \sum_{j=1}^{n} Y_j^*(h)$ exists and is finite] contains the set $[L_j \text{ i, } O(j)]^c$. Hence $\sigma([h| \lim_{n\to\infty} \sum_{j=1}^{n} Y_j^*(h)$ exists and is finite$]) = 1$. By Lemma 2-5 we have

$$\sigma([h| \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{n} a_j\, Y_j^*(h) = 0])$$

$$= \sigma([h| \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{n} Y_j(h) = 0]) = 1.$$

Based on Theorem 2-1 in [2], a simple proof of Theorem 4-2 is as follows.

Since $\sigma(\frac{Y_n}{a_n}) = 0$ for all $n = 1, 2, 3\ldots$ and $\sum_{n=1}^{\infty}\sigma(\frac{Y_n^2}{a_n^2}) < \infty$, $\{\sum_{j=1}^{n} \frac{Y_j}{a_j} \,|n = 1, 2, \ldots\}$ is a fundamental sequence in σ-probability.

Hence, by Theorem 2-1 of [2], we have

$$\sigma([h| \lim_{n\to\infty} \sum_{j=1}^{n} \frac{Y_j(h)}{a_j} \text{ exists and is finite}]) = 1.$$

By Lemma 2-5, we have

$$\sigma([h| \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{n} a_j \frac{Y_j(h)}{a_j} = \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{n} Y_j(h) = 0]) = 1 .$$

Corollary 4-2.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, $\{Y_n | n = 1, 2,\ldots\}$ is a sequence of coordinate mappings on H. Suppose that $\sigma(Y_n) = 0$ for all $n = 1, 2,\ldots,$ and $\sigma(Y_n^2) < \infty$, $\sigma(Y_n^2) = O(n^\theta)$ $n = 1, 2,\ldots$. Then, $\sigma([h| \lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{j=1}^{n} Y_j(h)]) = 1$, where $\theta \geq -1$, $\alpha > \frac{1+\theta}{2}$.

Proof: In Theorem 4-2, let $a_n = n^\alpha$. Since $\theta \geq -1$, $\alpha > \frac{1+\theta}{2} \geq 0$, $0 < a_1 \leq a_2 \leq \ldots$ and $\lim_{n\to\infty} = \infty$. By Theorem 4-2, we have Corollary 4-2.

Corollary 4-3.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H, $\{Y_n | n = 1, 2,\ldots\}$ is a sequence of coordinate mappings on H. Suppose that $\sigma(Y_n) = 0$, $\sigma(Y_n^2) \leq K < \infty$ for all $n = 1, 2,\ldots$. Then,

$$\sigma([h| \lim_{n\to\infty} \frac{S_n(h)}{n^\alpha} = 0]) = 1 \text{ for any } \alpha > \frac{1}{2} .$$

Proof: In Theorem 4-2, let $a_n = n^\alpha$. If $\alpha > \frac{1}{2}$, then $0 < a_1 < a_2 < \ldots$ and $\lim_{n\to\infty} a_n = \infty$. By Theorem 4-2, we have Corollary 4-3.

The next Theorem is the finitely additive version of Kolmogorov strong law of large numbers for an independent identically distributed strategy and a sequence of identical, coordinate mappings. Before proving the theorem, we need a lemma.

Lemma 4-1.

Suppose that $\sigma = \gamma \times \gamma \times \dots$ is an independent, identically distributed strategy on H and $\{Y_n | n = 1, 2, \dots\}$ is a sequence of identical, coordinate mappings defined on H. Suppose that f is a real-valued function defined on X such that $f(x_1) = Y(h)$ if $h = (x_1, x_2, \dots) \in H$. Then, $Y_1, Y_2, \dots$ are σ-integrable if and only if f is γ-integrable. Futhermore, $\sigma(Y_1) = \sigma(Y_2) = \dots = \gamma(f)$ whenever these integrals exist.

Proof: Since, for all $n = 1, 2, \dots,$ $f(x_n) = Y_n(h)$ if $h = (x_1, x_2, \dots, x_n, \dots)$ in H. Y_n is σ-integrable if and only if f is γ-integrable and $\sigma(Y_n) = \gamma(f)$ (by Lemma 2-1). Hence $\sigma(Y_1) = \sigma(Y_2) = \dots = \gamma(f)$.

Theorem 4-3.

Suppose that σ, $\{Y_n | n = 1, 2, \dots\}$, and f are as defined in Lemma 4-1. Then,

$$\sigma([h| \lim_{n \to \infty} \frac{S_n(h)}{n} = \mu]) = 1$$

if and only if f is γ-integrable and $\gamma(f) = \mu$ where μ is a real number.

Proof: First, suppose that f is γ-integrable and $\gamma(f) = \mu$. We can and do assume that $\mu = 0$. For each $n = 1, 2, 3, \dots,$ define a real-valued function Y_n^* on H, and a real-valued function

f_n^* on X by $Y_n^*(h) = Y_n(h)$ if $|Y_n(h)| \le n$

$= 0$ if $|Y_n(h)| > n$

$Y_n^*(h) = f_n^*(x_n)$ if $h = (x_1, x_2, \ldots, x_n, \ldots)$. And, notice that Y_n^* is σ-integrable, f_n^* is γ-integrable and $\sigma(Y_n^*) = \gamma(f_n^*)$.

Since $|f_n^*| \le |f|$ for all $n = 1, 2, \ldots$ and $\lim_{n\to\infty} f_n^* = f$ in γ-probability Dominated onvergence theorem), we have

$\lim_{n\to\infty} \gamma(f_n^*) = \gamma(f) = 0.$

Set $Z_n^* = Y_n^* - a_n$, $n = 1, 2, \ldots,$ where $a_n = \gamma(f_n^*) = \sigma(Y_n^*)$.

$$\sum_{n=1}^{\infty} \frac{\sigma(Z_n^{*2})}{n^2} \le \sum_{n=1}^{\infty} \frac{\sigma(Y_n^{*2})}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \int Y_n^2 \chi_{[|Y_n| \le n]} \, d\sigma .$$

Notice that

$$\int Y_n^2 \chi_{[|Y_n| \le n]} \, d\sigma = \int f_n^{*2} \, d\gamma$$

$$= \sum_{j=0}^{n-1} \int_{[j<|f_n^*|\le j+1]} f_n^{*2} \, d\gamma \le \sum_{j=0}^{n-1} (j+1)^2 \gamma(\{x | j < |f_n^*(x)| \le j+1\})$$

$$= \sum_{j=0}^{n-1} (j+1)^2 \gamma(\{x | j < |f(x)| \le j+1\}) .$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sigma(|Y_n^*|^2) \le \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{(j+1)^2}{n^2} \gamma(\{x|\, j <|f(x)| \le j+1\})$$

$$= \sum_{K=1}^{\infty}\sum_{n=k}^{\infty} \frac{K^2}{n^2} \gamma(\{x|K-1 <|f(x)| \le K\})$$

$$\le \sum_{K=1}^{\infty} K^2 \cdot \frac{2}{K} \gamma(\{x|K-1 <|f(x)| \le K\})$$

$$= \sum_{K=1}^{\infty} 2K\ \gamma(\{x|K-1 < |f(x)| \le K\})$$

$$= \sum_{K=1}^{\infty} 2(K-1)\ \gamma(\{x|K-1 < |f(x)| \le K\})$$

$$+ \sum_{K=1}^{\infty} 2\ \gamma(\{x|K-1 < |f(x)| \le K\})$$

$$\le 2\gamma(|f|) + 2\gamma(\{x|\ |f(x)| > 0\}) < \infty .$$

(The last step is implied by Theorem 3-2 in [3]).

By Theorem 4-1, we have

$$\sigma([h|\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Z_j^*(h) = 0]) = 1 \quad \text{since} \quad \lim_{n\to\infty} a_n = \gamma(f) = 0,$$

we have $\lim_{n\to\infty} \dfrac{\sum_{j=1}^{n} a_j}{n} = 0$. Hence

$$\sigma([h|\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j^*(h)]) = 1 .$$

For each $n = 1, 2, 3, \ldots$ set $K_n = [Y_n^* \neq Y_n]$, $K_n^* = \{h | h = (x_1, x_2, \ldots) | \ |f(x_n)| > n\}$. It is trivial that $\sigma(K_n) = \sigma(K_n^*)$ and $\sigma(K_n^*) = \gamma(\{x| \ |f(x)| > n\})$.

By Theorem 3-2 in [3],

$$\sum_{n=1}^{\infty} \sigma(K_n^*) = \sum_{n=1}^{\infty} \gamma(\{x| \quad |f(x)| > n\})$$

$$\leq \sum_{n=0}^{\infty} \gamma(\{x| \quad |f(x)| > n\}) \leq \gamma(|f|) + 1 < \infty .$$

By Corollary 3-1, we have

$$\sigma([K_n \text{ i, } O(n)]^c) = 1$$

since

$$[K_n \text{ i, } O(n)]^c \cap [h| \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j^* (h) = 0]$$

$$\subseteq [h| \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0] ,$$

$$\sigma([h| \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0]) = 1 .$$

Next, suppose that $\sigma([h| \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0]) = 1$ then,

$$\sigma([h| \lim_{n\to\infty} \frac{1}{n-1} \sum_{j=1}^{n-1} Y_j(h) = 0]) = 1 \quad \text{and hence}$$

$$\sigma([h| \lim_{n\to\infty} \frac{Y_n(h)}{n} = 0]) = 1 .$$

Let, for each $j = 1, 2, 3, \ldots$ $K_j = [h | h \in H | \frac{Y_j(h)}{j}| > 1]$.

If $\sum_{j=1}^{\infty} \sigma(K_j) = \infty$, then, by Corollary 3-2, $\sigma([K_n \text{ i, } O(n)]) = 1$.

Hence, it is necessary that $\sum_{n=1}^{\infty} \sigma(K_n) < \infty$. Let $K_n^* = [x| \; |f(x)| > n]$.

Then $\sigma(K_n) = \gamma(K_n^*)$

$$\sum_{n=1}^{\infty} \sigma(K_n) = \sum_{n=1}^{\infty} \gamma(K_n^*) < \infty \; .$$

By Theorem 3-2 in [3], we have f is γ-integrable. Hence $Y_1, Y_2, \ldots$ are σ-integrable. By the first part of the proof above, we should have

$$\int f d\gamma = \int Y_1 d\sigma = \int Y_2 \, d\sigma = \ldots = 0 \; .$$

The proof of Theorem 4-3, now, is complete.

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